# ON THE LAPLACE-POISSON MIXED EQUATION

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I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY Raymond F. Borden

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# ON THE LAPLACE-POISSON MIXED EQUATION.

## INTRODUCTION.

We designate as the Laplace-Poisson mixed equation, the equation*

$$(1) \qquad f'(x+1)+p(x)f'(x)+q(x)f(x+1)+m(x)f(x) = 0,$$

which was first studied by Poisson†, and which is analogous in form and in theory to the Laplace partial differential equation

$$s + ap + bq + cz = 0.$$

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* The coefficients $p(x)$, $q(x)$, and $m(x)$ are analytic functions of the real or complex variable x.

† Jour. de l'École Polytechnique: t.6, (1806), pp.127-141. See also Lacroix, "Traité du Calcul", 3rd. Ed., Vol.3, pp.575-600, for the work of Poisson and other early investigators in the field. Other papers on mixed equations are the following:- Vernier: Ann. de Math. 13, (1882), pp.258-267; Gregory: Cambridge Math. Jour. 1, (1839), p.54; Boole: "A Treatise on the Calculus of Finite Differences", (1860); Walton: Quart. Jour. 10, (1870), 248-253; Combescure: Ann. Éc. Nor. Sup. (2)3, (1874), 305-362; Laurent: "Traité de Analyse", (1890), Vol.6, pp.234-236; Lemeray: Edinburgh Math. Soc. Proc., (1898), 13-14; Oltramare: Assoc. Fr. Marseille, 20, (1891), 66-82; Oltramare: Bordeaux Assoc. Fr. Bull. 24, (1899), 175-186; Oltramare: "Calcul de Generalisation", (1899); Brajtzew: Moscow Coll., (1901) (2 articles); Pincherle: Rendiconti Pal. 18, (1904); Pincherle: Mem. Soc. Italiana d. Sc. (3), 15, (1907); Meissner: Schweiz. Bauzeitung, 54; Polussuchin: Zurich Diss., (1910); Schmidt: Math. Ann. 70 (1911), 499-524; Bateman: Proc. 5th. Int. Cong. Math., (1912), 291-294; Haag: Bull. de Sc. Math., 36, (1912), 10-24; Schürer: Ber. Gew. Wiss. Leipzig, (1912), 167-236; (1913), 139-143; (1914), 137-159; Carmichael: Am. Jour. 35, (1913), 151-162; Cesàro: Nouv. Ann. (3)4, (1913), 36-41; Bennett: Ann. of Math. (2)18, 24-30.

Poisson finds solutions in finite form by means of transformations analogous to those used by Laplace in solving the differential equation written above. These transformations put the mixed equation into equations of the same form, viz:

$$F'(x+1)+P(x)F'(x)+Q(x)F(x+1)+M(x)F(x) = 0.$$

When certain relations exist between the coefficients of one of the transformed equations, Poisson solves that equation by the standard methods of solving linear difference and differential equations of the first order, and then obtains the solution of the original equation by reversing the transformations.

The remark of Poisson, that the theory of this type of equation is but little advanced, still holds true more than a century later. In this paper the elementary theory of the equation is extended along the lines initiated by Poisson. For the sake of completeness and clearness the work of Poisson is largely included, but from a slightly different point of view; the fundamental invariants of the equation being emphasized. The theory of the invariants under the group of transformations $f(x) = v(x)g(x)$ is developed along the same lines as is the corresponding theory of the Laplace equation.* Largely the same methods are used, the analogy being very close. The results are summarized below by sections.

I. The functions

$$\frac{m(x)}{p(x)} - q(x) - \frac{p'(x)}{p(x)}, \text{ and } \frac{m(x)}{p(x)} - q(x-1)$$

form a fundamental set of invariants under the group of transform-

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*An exposition of the theory of the partial differential equation $s + ap + bq + cz = 0$ may be found in Forsyth's "Theory of Differential Equations", Vol.6, pp.44-96.

ations $f(x) = v(x)g(x)$, which transformations do not change the form of the equation.

II. When one of the fundamental invariants is zero the equation is of such nature that it may be obtained by differentiating a difference equation or else by applying the difference operation to a differential equation. That is it may be solved by integrating first a linear differential [difference] equation and then a linear difference [differential] equation. These solutions each involve an arbitrary constant and an arbitrary periodic function of period one.

III. The Laplace-Poisson transformations

$$S)\quad f_{S_1}(x) = f(x+1) + p(x)f(x) \quad \text{and} \quad T)\quad f_{T_1}(x) = f'(x) + q(x-1)f(x)$$

leave the form of the equation unchanged. The invariants of the equation gotten by applying S or T are simply expressible in terms of the invariants of the original equation. The two transformations are, in a sense, inverses of each other; for the application of both in either order to (1) gives an equation with the same invariants as (1). Successive applications of S, or of T, give equations of the same type whose invariants can be expressed in terms of the invariants of the preceding equations under the successive transformations, and therefore in terms of the invariants of the original equation.

IV. The solutions of the equations obtained by successive applications of S or T may be obtained in terms of the solution of the *n*th transformed equation and the invariants of the intermediate equations. In particular, the solution of the original equation may be thus obtained.

V. The term <u>rank</u> of the equation is introduced in accordance with the nomenclature of the corresponding theory of the Laplace partial differential equation.* The mixed equation is said to be of finite rank when a finite number of applications of S, or of T, results in an equation with a vanishing invariant. The equation can then be solved in finite form, and an arbitrary part of the solution can be so chosen that quadratures of arbitrary functions are not involved. The equation is said to be of rank n+1 of the first kind when n applications of S give an equation with a vanishing invariant. This is a necessary and sufficient condition for a solution of the form

$$f(x) = E_0(x)F(x) + E_1(x)F'(x) + \text{--------} + E_n(x)F^{(n)}(x),$$

where the E's are determinate functions and F(x) is an arbitrary periodic function of period one. The mixed equation is said to be of rank n+1 of the second kind when n applications of T give an equation with a vanishing invariant. In this case, the solution without quadratures of arbitrary functions is a determinate function of x multiplied by an arbitrary constant.

<u>VI, VII.</u> The restrictions on the coefficients of the mixed equation in order that it be of finite rank of the first kind or of the second kind are found.

<u>VIII.</u> When the equation is of finite rank with respect to either S or T, it is said to be of doubly finite rank. The restrictions on the coefficients of such an equation are found.

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* Forsyth, l.c., p. 60.

IX. Generalizations of the Laplace-Poisson transformations analogous to the transformations used by Lévy* in connection with the Laplace differential equation are here tried with a result similar to that found by Lévy, viz: that they are not, in general, useful in obtaining an equation of the type (1) with a vanishing invariant.†

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*Jour. de l'École Polytech. t.38, (1886), p.67.

†See Forsyth, l.c., p.94.

## I. THE INVARIANTS OF THE EQUATION.

The equation*

$$(1)\qquad f'(x+1)+p(x)f'(x)+q(x)f(x+1)+m(x)f(x)=0$$

is put by a transformation of the group

$$f(x) = v(x)g(x)$$

into the form

$$(2)\qquad g'(x+1)+P(x)g'(x)+Q(x)g(x+1)+M(x)g(x)=0,$$

where

$$P(x) = p(x)\frac{v(x)}{v(x+1)}, \qquad Q(x) = \frac{v'(x+1)}{v(x+1)}+q(x),$$

and

$$M(x) = m(x)\frac{v(x)}{v(x+1)} + p(x)\frac{v'(x)}{v(x+1)}.$$

The form of the equation is therefore unaltered by the substitution. By eliminating $v(x)$ in two ways from the relations between the coefficients of (1) and (2), we may obtain

$$p(x)\left[M(x)-P(x)Q(x)-P'(x)\right] = P(x)\left[m(x)-p(x)q(x)-p'(x)\right],$$

and

$$p(x)\left[M(x)-P(x)Q(x-1)\right] = P(x)\left[m(x)-p(x)q(x-1)\right].$$

Hence

$$I(x) = \frac{m(x)}{p(x)} - q(x) - \frac{p'(x)}{p(x)}$$

and

$$J(x) = \frac{m(x)}{p(x)} - q(x-1)$$

are absolute invariants of the equation (1) under the group of

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* We shall develop the theory only for the case when $p(x)$ is not zero. When $p(x)=0$, $I(x)$ and $J(x)$ are illusory and $\alpha(x)$ and $\beta(x)$ are each equal to $m(x)$. It may be readily seen by following through this paper that the case $p(x)=0$ can be carried, but the conditions for solutions here developed become simply $m(x)=0$ when $p(x)=0$, and the equation is then not a true mixed equation.

transformations $f(x) = v(x)g(x)$. We shall find it convenient to use also the relative invariants

$$\alpha(x) = m(x)-p(x)q(x)-p'(x) \text{ and } \beta(x) = m(x)-p(x)q(x-1).$$

These functions are each multiplied by $v(x)/v(x+1)$ at each application of $f(x) = v(x)g(x)$.

We will now show that $I(x)$ and $J(x)$ form a fundamental set of invariants of the equation (1);i.e. that all invariants of (1) under $f(x) = v(x)g(x)$ can be expressed as functions of $I(x)$ and $J(x)$ involving only algebraic operations,the operations of the differential calculus and the difference calculus,and their inverses.

We will choose $v(x)$ in the transformation $f(x) = v(x)g(x)$ so that the equation (1) will be put into the form (2) subject to the restriction

$$P(x)Q(x) = M(x).$$

This condition reduces to

$$\frac{v(x)}{v(x+1)}\left[p(x)q(x) - m(x)\right] = p(x)\frac{d}{dx}\left[\frac{v(x)}{v(x+1)}\right],$$

whence we may take

$$\frac{v(x)}{v(x+1)} = e^{\int_{x_0}^{x}\frac{p(x)q(x) - m(x)}{p(x)}dx}.$$

This condition is also sufficient. So,by choosing $v(x)$ properly, we can transform the equation (1) into the form

$$g'(x+1) + P(x)g'(x) + Q(x)g(x+1) + P(x)Q(x)g(x) = 0.$$

The I and J invariants of this equation are

$$I(x) = \frac{-P'(x)}{P(x)},$$

and $$J(x) = Q(x) - Q(x-1) = \Delta Q(x-1).$$

Accordingly

$$P(x) = c e^{-\int_{x_0}^{x} I(x)dx},$$

and

$$Q(x) = \sum J(x+1),$$

where $\sum$ denotes some particular finite integral. Hence the transformed equation is

$$(3)\qquad g'(x+1) + c e^{-\int_{x_0}^{x} I(x)dx} g'(x) + \sum J(x+1)\, g(x+1) + c e^{\int_{x_0}^{x} I(x)dx} \sum J(x+1)\, g(x) = 0.$$

This is of the same form as the original equation (1), and is derived from (1) by a transformation of the group $f(x) = v(x)g(x)$. Therefore (3) has the same invariants as (1) under transformations of the type considered. Since the invariants are functions of the coefficients alone, it follows that all the invariants of (3) are expressible in terms of $I(x)$ and $J(x)$ only. We shall refer to $I(x)$ and $J(x)$ simply as the invariants of the equation.

## II. SOLUTIONS WHEN ONE INVARIANT IS ZERO.

If $l(x)=0$, then $\alpha(x)=0$,

and $$m(x) = p'(x)+p(x)q(x).$$

The equation may then be written in the form

$$\frac{d}{dx}\Big[f(x+1)+p(x)f(x)\Big]+q(x)\Big[f(x+1)+p(x)f(x)\Big]=0,$$

whence

$$f(x+1)+p(x)f(x) = ce^{-\int_{x_0}^{x}q(x)dx}.$$

To solve this, we first solve the homogeneous equation

$$g(x+1) - \Big[-p(x)\Big]g(x) = 0$$

as follows:

$$\log g(x+1) - \log g(x) = \log\Big[-p(x)\Big],$$

or

$$g(x) = \varphi(x)\, e^{\sum \log[-p(x)]}$$

where $\varphi(x)$ is an arbitrary periodic function of period one, and $\sum$ denotes a finite integral* for some range of the variable x.

Let $f(x)=u(x)g(x)$ and substitute in the non-homogeneous equation. We get, if we take $\varphi(x)=1$,

$$u(x+1)\, e^{\sum\log[-p(x+1)]} - \Big[-p(x)\Big]u(x)\, e^{\sum\log[-p(x)]} = ce^{-\int_{x_0}^{x}q(x)dx}.$$

Hence $$u(x+1)-u(x) = ce^{-\int_{x_0}^{x}q(x)dx - \sum\log[-p(x+1)]},$$

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* F(x) is said to be a finite integral of G(x) if

$$F(x+1) - F(x) = G(x).$$

In this paper, the symbol $\sum$ without limits of summation denotes a finite integral. When used with limits, e.g., $\sum_{K=0}^{n}$, it denotes an ordinary summation.

and therefore

$$u(x) = \sum\left[ce^{-\int_{x_0}^{x} q(x)dx - \sum\log[-p(x+1)]}\right] + F(x),$$

where $F(x)$ has the period one and is otherwise arbitrary.

So we have

$$(4) \qquad f(x) = e^{\sum\log[-p(x)]}\left\{F(x)+\sum ce^{-\int_{x_0}^{x} q(x)dx - \sum\log[-p(x+1)]}\right\}.$$

If $J(x)=0$, then $\beta(x)=0$,

and $\qquad m(x) = p(x)q(x-1)$.

The equation may then be written

$$f'(x+1)+q(x)f(x+1)+p(x)\left[f'(x)+q(x-1)f(x)\right] = 0,$$

from which we obtain*

$$\Delta\log\left[f'(x)+q(x-1)f(x)\right] = \log\left[-p(x)\right],$$

whence

$$f'(x)+q(x-1)f(x) = \theta(x)\, e^{\sum\log[-p(x)]},$$

$\theta(x)$ being an arbitrary periodic function of period one. Hence we have

$$(5) \quad f(x) = e^{-\int_{x_0}^{x} q(x-1)dx}\int_{x_0}^{x}\theta(x)\, e^{\sum\log[-p(x)]+\int_{x_0}^{x} q(x-1)dx} + Ke^{-\int_{x_0}^{x} q(x-1)dx},$$

where $K$ is an arbitrary constant.

We have thus shown that when $I(x)=0$ $[J(x)=0]$ a solution of the equation (1) can be obtained in finite form by solving, first a linear differential [difference] equation of the first order, and then a linear difference [differential] equation of the first order.

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* $\Delta$ denotes the difference of a function, i.e.,

$$\Delta v(x) = v(x+1) - v(x).$$

## III. THE LAPLACE-POISSON TRANSFORMATIONS AND THE INVARIANTS OF THE RESULTING EQUATIONS.

The Laplace-Poisson transformation

$$(S)\qquad f_{S_1}(x) = f(x+1) + p(x)f(x)$$

transforms the equation (1) into an equation of the same form, viz:

$$(6)\qquad f'_{S_1}(x+1) + p_{S_1}(x)f'_{S_1}(x) + q_{S_1}(x)f_{S_1}(x+1) + m_{S_1}(x)f_{S_1}(x) = 0,$$

where

$$p_{S_1}(x) = p(x)\frac{\alpha(x+1)}{\alpha(x)} = p(x+1)\frac{I(x+1)}{I(x)},$$

$$q_{S_1}(x) = q(x+1),$$

and

$$m_{S_1}(x) = p(x+1)q(x)\frac{I(x+1)}{I(x)} + p(x+1)I(x+1).$$

The invariants of (6) under the group $f_{S_1}(x) = v(x)g(x)$ are

$$J_{S_1}(x) = \frac{m_{S_1}(x)}{p_{S_1}(x)} - q_{S_1}(x-1)$$

$$= q(x) + I(x) - q(x)$$

$$= I(x).$$

and

$$I_{S_1}(x) = \frac{m_{S_1}(x)}{p_{S_1}(x)} - q_{S_1}(x) - \frac{p'_{S_1}(x)}{p_{S_1}(x)}$$

$$= q(x) + I(x) - q(x+1) - \frac{p'(x+1)}{p(x+1)} - \Delta\frac{d}{dx}\log I(x).$$

If we add and subtract $m(x+1)/p(x+1)$, we get

$$I_{S_1}(x) = I(x+1) + I(x) - J(x+1) - \Delta\frac{d}{dx}\log I(x).$$

Under the Laplace-Poisson transformation

(T) $f_{T_1}(x) = f'(x) + q(x-1)f(x)$

the equation (1) becomes

(7) $$f'_{T_1}(x+1) + p_{T_1}(x)f'_{T_1}(x) + q_{T_1}(x)f_{T_1}(x+1) + m_{T_1}(x)f_{T_1}(x) = 0,$$

in which the coefficients may be reduced to the following forms:

$$p_{T_1}(x) = p(x),$$

$$q_{T_1}(x) = q(x-1) + \frac{J'(x)}{J(x)} + \frac{p'(x)}{p(x)},$$

and $$m_{T_1}(x) = p(x)J(x) + p(x)q(x-1) - p(x)\frac{J'(x)}{J(x)}.$$

The invariants of (7) under the group $f_{T_1}(x) = v(x)g(x)$ are

$$I_{T_1}(x) = \frac{m_{T_1}(x)}{p_{T_1}(x)} - q_{T_1}(x) - \frac{p'_{T_1}(x)}{p_{T_1}(x)}$$

$$= J(x) + q(x-1) - \frac{J'(x)}{J(x)} - q(x-1) + \frac{p'(x)}{p(x)} + \frac{J'(x)}{J(x)} - \frac{p'(x)}{p(x)}$$

$$= J(x).$$

and $$J_{T_1}(x) = \frac{m_{T_1}(x)}{p_{T_1}(x)} - q_{T_1}(x-1)$$

$$= J(x) + q(x-1) - \frac{J'(x)}{J(x)} + \frac{J'(x-1)}{J(x-1)} + \frac{p'(x-1)}{p(x-1)} - q(x-2).$$

If we add and subtract $m(x-1)/p(x-1)$, we get

$$J_{T_1}(x) = J(x) + J(x-1) - I(x-1) - \Delta\frac{d}{dx}\log J(x-1).$$

Hence we see that the transformations S and T each transform the equation (1) into an equation of the same form, the invariants of which may be simply expressed in terms of the invariants of (1).

The two transformations S and T are, in a sense, inverses of each other; for TS gives

$$
\begin{aligned}
f_{TS}(x) &= f'_{S_1}(x) + q(x) f_{S_1}(x) \\
&= f'(x+1) + p(x)f'(x) + p'(x)f(x) + q(x)f(x+1) \\
&\qquad + p(x)q(x)f(x) \\
&= \left[p'(x) + p(x)q(x) - m(x)\right] f(x) \\
&\qquad -p(x)I(x)f(x) = -\alpha(x)f(x),
\end{aligned}
$$

and ST gives

$$
\begin{aligned}
f_{ST}(x) &= f_{T_1}(x+1) + p(x) f_{T_1}(x) \\
&= f'(x+1) + q(x)f(x+1) + p(x)f'(x) + p(x)q(x-1)f(x) \\
&= \left[p(x)q(x-1) - m(x)\right] f(x) \\
&\qquad -p(x)J(x)f(x) = -\beta(x)f(x).
\end{aligned}
$$

Hence the equations resulting from applications of TS and ST have the same invariants as has the original equation. Furthermore we may transform the equation (1) into itself as follows. Apply TS[ST]. The resulting equation is that obtained by replacing $f(x)$ in (1) by $\alpha(x)f(x)$ $[\beta(x)f(x)]$. Then the transformation $f(x) = g(x)/\alpha(x)$ $[f(x) = g(x)/\beta(x)]$ brings us back to the equation (1).

Let $I_{S_n}(x)$ and $J_{S_n}(x)$ be the invariants of the equation obtained by n successive applications of S. Then we have

$$J_{S_n}(x) = I_{S_{n-1}}(x),$$

and

$$I_{S_n}(x) = I_{S_{n-1}}(x+1) + I_{S_{n-1}}(x) - J_{S_{n-1}}(x) - \Delta \frac{d}{dx} \log I_{S_{n-1}}(x).$$

So we can write

$$
\begin{aligned}
I_{S_n}(x) - I_{S_{n-1}}(x+1) - I_{S_{n-1}}(x) + I_{S_{n-2}}(x+1) &= -\Delta \frac{d}{dx} \log I_{S_{n-1}}(x) \\
I_{S_{n-1}}(x) - I_{S_{n-2}}(x+1) - I_{S_{n-2}}(x) + I_{S_{n-3}}(x+1) &= -\Delta \frac{d}{dx} \log I_{S_{n-2}}(x) \\
\text{-----------------------} & \\
\text{-----------------------} & \\
I_{S_1}(x) - I(x+1) - I(x) + J(x+1) &= -\Delta \frac{d}{dx} \log I(x).
\end{aligned}
$$

Adding these, we get

$$I_{S_n}(x) - I_{S_{n-1}}(x+1) = I(x) - J(x+1) - \Delta\frac{d}{dx}\log\left[I(x)I_{S_1}(x) ---- I_{S_{n-1}}(x)\right].$$

Write this for n, n-1, n-2, ------ successively, and add 1 to the argument at each step. We have then

$$I_{S_n}(x) - I_{S_{n-1}}(x+1) = I(x) - J(x+1) - \Delta\frac{d}{dx}\log\left[I(x)I_{S_1}(x) --- I_{S_{n-1}}(x)\right]$$

$$I_{S_{n-1}}(x+1) - I_{S_{n-2}}(x+2) = I(x+1) - J(x+2) - \Delta\frac{d}{dx}\log\left[I(x+1) --- I_{S_{n-2}}(x\ 1)\right]$$

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$$I_{S_1}(x+n-1) - I(x+n) = I(x+n-1) - J(x+n) - \Delta\frac{d}{dx}\log I(x+n-1).$$

Adding, we get

$$I_{S_n}(x) - I(x+n) = \sum_{k=0}^{n-1} I(x+k) - \sum_{k=1}^{n} J(x+k)$$
$$- \Delta\frac{d}{dx}\log\left[\prod_{k=0}^{n-1} I(x+k)\prod_{k=0}^{n-2} I_{S_1}(x+k) ----- I_{S_{n-1}}(x)\right],$$

or

$$I_{S_n}(x) = I(x) + \sum_{k=1}^{n}\left[I(x+k) - J(x+k)\right] \tag{8}$$
$$- \Delta\frac{d}{dx}\log\left[\prod_{k=0}^{n-1} I(x+k)\prod_{k=0}^{n-2} I_{S_1}(x+k)\prod_{k=0}^{n-3} I_{S_2}(x+k) ---- I_{S_{n-1}}(x)\right].$$

Also

$$J_{S_n}(x) = I_{S_{n-1}}(x)$$
$$= I(x) + \sum_{k=1}^{n-1}\left[I(x+k) - J(x+k)\right] \tag{9}$$
$$- \Delta\frac{d}{dx}\log\left[\prod_{k=0}^{n-2} I(x+k)\prod_{k=0}^{n-3} I_{S_1}(x+k) ------------ I_{S_{n-2}}(x)\right].$$

If $I_{T_n}(x)$ and $J_{T_n}(x)$ are the invariants of the equation obtained by applying T n times, we have

$$I_{T_n}(x) = J_{T_{n-1}}(x),$$

and

$$J_{T_n}(x) = J_{T_{n-1}}(x) + J_{T_{n-1}}(x-1) - I_{T_{n-1}}(x-1) - \Delta\frac{d}{dx}\log J_{T_{n-1}}(x-1).$$

So we may write

$$J_{T_n}(x)-J_{T_{n-1}}(x-1)-J_{T_{n-1}}(x)+J_{T_{n-2}}(x-1) = -\Delta\frac{d}{dx}\log J_{T_{n-1}}(x-1)$$

$$J_{T_{n-1}}(x)-J_{T_{n-2}}(x-1)-J_{T_{n-2}}(x)+J_{T_{n-3}}(x-1) = -\Delta\frac{d}{dx}\log J_{T_{n-2}}(x-1)$$

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$$J_{T_1}(x)-J(x-1)-J(x)+I(x-1) = -\Delta\frac{d}{dx}\log J(x-1).$$

Adding these, we get

$$J_{T_n}(x)-J_{T_{n-1}}(x-1) = J(x)-I(x-1)-\Delta\frac{d}{dx}\log\left[J(x-1)J_{T_1}(x-1)----J_{T_{n-1}}(x-1)\right].$$

Write this for n, n-1, n-2, ---------- successively, subtracting 1 from the argument at each step. We have then

$$J_{T_n}(x)-J_{T_{n-1}}(x-1) = J(x)-I(x-1)-\Delta\frac{d}{dx}\log\left[J(x-1)J_{T_1}(x-1)------J_{T_{n-1}}(x-1)\right]$$

$$J_{T_{n-1}}(x-1)-J_{T_{n-2}}(x-2) = J(x-1)-I(x-2)-\Delta\frac{d}{dx}\log\left[J(x-2)J_{T_1}(x-2)-----J_{T_{n-2}}(x-2)\right]$$

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$$J_{T_1}(x-n+1)-J(x-n) = J(x-n+1)-I(x-n)-\Delta\frac{d}{dx}\log J(x-n)$$

Adding these, we get

$$J_{T_n}(x) - J(x-n) = \sum_{k=0}^{n-1} J(x-k) - \sum_{k=1}^{n} I(x-k) - \Delta\frac{d}{dx}\log\left[\prod_{k=1}^{n} J(x-k)\prod_{k=1}^{n-1} J_{T_1}(x-k)-----J_{T_{n-1}}(x-1)\right],$$

and therefore

$$(10)\qquad J_{T_n}(x) = J(x) + \sum_{k=1}^{n}\left[J(x-k) - I(x-k)\right] - \Delta\frac{d}{dx}\log\left[\prod_{k=1}^{n} J(x-k)\prod_{k=1}^{n-1} J_{T_1}(x-k)----J_{T_{n-1}}(x-1)\right].$$

Also

$$(11)\qquad I_{T_n}(x) = J_{T_{n-1}}(x) = J(x) + \sum_{k=1}^{n-1}\left[J(x-k) - I(x-k)\right] - \Delta\frac{d}{dx}\log\left[\prod_{k=1}^{n-1} J(x-k)\prod_{k=1}^{n-2} J_{T_1}(x-k)----J_{T_{n-2}}(x-1)\right].$$

So we have the result that after n successive transformations of the equation (1) by S[T], we arrive at an equation whose invariants can be expressed explicitly in terms of the invariants of (1) and the invariants of the intermediate equations obtained by 1,2,3,------,n-1 applications of S[T], and hence in terms of the invariants of the original equation.

## IV. SOLUTIONS OF SUCCESSIVELY TRANSFORMED EQUATIONS.

After $n+1$ applications of S the new dependent variable is $f_{S_{n+1}}(x)$. Operate with T and call the resulting dependent variable $f_{S_{n+1},T_1}(x)$. We have then

$$f_{S_{n+1}}(x) = f_{S_n}(x+1) + p_{S_n}(x) f_{S_n}(x)$$

$$f_{S_{n+1},T_1}(x) = f'_{S_{n+1}}(x) + q_{S_{n+1}}(x-1) f_{S_{n+1}}(x)$$

$$= -p_{S_n}(x) I_{S_n}(x) f_{S_n}(x).$$

Multiplying by $\exp\left[\int_{x_0}^{x} q(x+n)dx\right]$, and remembering that $q_{S_{n+1}}(x-1) = q(x+n)$, we have

$$\frac{d}{dx}\left[f_{S_{n+1}}(x)\, e^{\int_{x_0}^{x} q(x+n)dx}\right] = -p_{S_n}(x) I_{S_n}(x) f_{S_n}(x)\, e^{\int_{x_0}^{x} q(x+n)dx},$$

and therefore

$$f_{S_n}(x)\, e^{\int_{x_0}^{x} q(x+n)dx} = \frac{-1}{p_{S_n}(x) I_{S_n}(x)} \frac{d}{dx}\left[f_{S_{n+1}}(x)\, e^{\int_{x_0}^{x} q(x+n)dx}\right].$$

Since

$$e^{\int_{x_0}^{x} q(x+n)dx} = e^{\int_{x_0}^{x}\left[q(x+n) - q(x+n-1) + q(x+n-1)\right]dx}$$

$$= e^{\int_{x_0}^{x} q(x+n-1)dx}\; e^{\int_{x_0}^{x}\Delta q(x+n-1)dx},$$

we may write

$$f_{S_n}(x)\, e^{\int_{x_0}^{x} q(x+n-1)dx} = A_n \frac{dB_n}{dx}$$

where

$$A_n = \frac{e^{-\int_{x_0}^{x}\Delta q(x+n-1)dx}}{-p_{S_n}(x) I_{S_n}(x)},$$

and $\quad B_n = f_{S_{n+1}}(x)\, e^{\int_{x_0}^{x} q(x+n)dx}.$

Then we have

$$f_{S_{n-1}}(x)\, e^{\int_{x_0}^{x} q(x+n-2)dx} = A_{n-1}\frac{d}{dx}\left(A_n \frac{d}{dx}B_n\right)$$

$$f_{S_{n-2}}(x)\, e^{\int_{x_0}^{x} q(x+n-3)dx} = A_{n-2}\frac{d}{dx}\left[A_{n-1}\frac{d}{dx}\left(A_n \frac{d}{dx} B_n\right)\right]$$

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$$(12) \qquad f(x)\, e^{\int_{x_0}^{x} q(x-1)dx} = A_0\frac{d}{dx}\left\{A_1\frac{d}{dx}\left[-------- \frac{d}{dx}\left(A_n \frac{d}{dx} B_n\right)\right]\right\},$$

where the successive A's are gotten by replacing n in $A_n$ by n-1, n-2,n-3,------- 2,1,0, if we agree that $p_{S_0}(x) = p(x)$, and $I_{S_0}(x) = I(x)$.

In a similar way we can get an expression for f(x) in terms of $f_{T_n}(x)$, the dependent variable after n applications of T. The (n+1)th application of T gives

$$f_{T_{n+1}}(x) = f'_{T_n}(x) + q_{T_n}(x-1)f_{T_n}(x).$$

Then, operating with S, we get, since $p_{T_{n+1}}(x) = p(x)$,

$$f_{T_{n+1},S_1}(x) = f_{T_{n+1}}(x+1) + p(x)f_{T_{n+1}}(x)$$
$$= -p(x)J_{T_n}(x)f_{T_n}(x).$$

We want to find a quantity $\chi(x)$ such that

$$\chi(x)\left[f_{T_{n+1}}(x+1) + p(x)f_{T_{n+1}}(x)\right] = \Delta\left[f_{T_{n+1}}(x)\,\psi(x)\right].$$

We may then take

$$\chi(x) = \psi(x+1),$$

and $\quad p(x)\chi(x) = -\psi(x).$

These two conditions give

$$\psi(x) = e^{-\sum \log\left[-p(x)\right]}.$$

Then we have

$$\chi(x) = \psi(x+1) = e^{-\sum \log\left(-p(x+1)\right)}.$$

So we write

$$\left[f_{T_{n+1}}(x+1) + p(x) f_{T_{n+1}}(x)\right] e^{-\sum \log\left[-p(x+1)\right]}$$

$$= -p(x) J_{T_n}(x)\, e^{-\sum \log\left[-p(x+1)\right]} f_{T_n}(x);$$

whence

$$\Delta\left\{f_{T_{n+1}}(x)\, e^{-\sum \log\left[-p(x)\right]}\right\} = -p(x) J_{T_n}(x) f_{T_n}(x)\, e^{-\sum \log\left[-p(x+1)\right]},$$

and therefore

$$f_{T_n}(x)\, e^{-\sum \log\left[-p(x+1)\right]} = C_n \Delta D_n,$$

where

$$C_n = \frac{-1}{p(x) J_{T_n}(x)},$$

and $$D_n = f_{T_{n+1}}(x)\, e^{-\sum \log\left[-p(x)\right]}.$$

Then we have

$$f_{T_{n-1}}(x)\, e^{-\sum \log\left[-p(x)\right]} = C_{n-1} \Delta(C_n \Delta D_n)$$

$$f_{T_{n-2}}(x)\, e^{-\sum \log\left[-p(x)\right]} = C_{n-2} \Delta\left[C_{n-1} \Delta(C_n \Delta D_n)\right]$$

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$$f(x)\, e^{-\sum \log\left[-p(x)\right]} = C_0 \Delta\left\{C_1 \Delta\left[\text{---------} \Delta(C_n \Delta D_n)\right]\right\} \tag{13}$$

where the C's are gotten by replacing n in $C_n$ by n-1, n-2, ----etc., if we agree that $J_{T_0}(x) = J(x)$.

Now we have expressed the solution f(x) of the equation (1) in terms of $f_{S_n}(x)$ $\left[f_{T_n}(x)\right]$, the solution of the nth transformed equation under S $[T]$. We have seen that we can find $f_{S_n}(x)$ $\left[f_{T_n}(x)\right]$ if $I_{S_n}(x) = 0$ $\left[I_{T_n}(x) = 0\right]$ or if $J_{S_n}(x) = 0$ $\left[J_{T_n}(x) = 0\right]$, and these solutions will be in finite form.

## V. THE RANK OF THE EQUATION.

Suppose $I_{S_n}(x) = 0$. The equation may then be written in the form

$$\frac{d}{dx}\left[f_{S_n}(x+1) + p_{S_n}(x)f_{S_n}(x)\right] + q_{S_n}(x)\left[f_{S_n}(x+1) + p_{S_n}(x)f_{S_n}(x)\right] = 0,$$

which, as may be seen from § II, has a solution of the form

$$f_{S_n}(x) = e^{\Sigma\log\left[-p_{S_n}(x)\right]}\left\{F(x) + \sum c e^{-\int_{x_0}^{x} q_{S_n}(x)dx - \Sigma\log\left[-p_{S_n}(x+1)\right]}\right\},$$

where $F(x)$ has the period one and is otherwise arbitrary, and where $c$ is an arbitrary constant.

As before, denote $e^{\int_{x_0}^{x} q(x+n-1)dx} f_{S_n}(x)$ by $B_{n-1}$. Then this expression is seen to be of the form

$$B_{n-1} = \eta(x)\left[F(x) + c\sum\xi(x)\right].$$

Making use of a previously derived formula, viz:

$$(12)\qquad e^{\int_{x_0}^{x} q(x-1)dx} f(x) = A_0\frac{d}{dx}\left\{A_1\frac{d}{dx}\left[\text{---------}\frac{d}{dx}\left(A_{n-1}\frac{d}{dx}B_{n-1}\right)\right]\right\},$$

we get an expression for $f(x)$ of the form

$$f(x) = E_0(x)\left\{F(x) + c\Sigma\xi(x)\right\} + E_1(x)\left\{F'(x) + c\left[\Sigma\xi(x)\right]'\right\}$$
$$+ E_2(x)\left\{F''(x) + \left[c\Sigma\xi(x)\right]''\right\} + \text{----------}$$
$$+ \text{-------} + E_n(x)\left\{F^{(n)}(x) + \left[c\ \Sigma\xi(x)\right]^{(n)}\right\}$$

where the E's are determinate functions of x.

Taking the particular integral for which $c = 0$, we have a simpler integral which does not involve $\Sigma\xi(x)$, viz:

$$(14)\qquad f(x) = E_0(x)F(x) + E_1(x)F'(x) + \text{-----------} + E_n(x)F^{(n)}(x).$$

We will call this an integral of <u>rank</u> $n+1$ of the <u>first kind</u>. We will call the original equation of rank $n+1$ of the first kind when $I_{S_n}(x) = 0$* and $I_{S_k}(x) \neq 0$, where $k$ takes the values $0,1,2,3,$---------, $n-1$.

Conversely, if the original equation

(1) $$f'(x+1)+p(x)f'(x)+q(x)f(x+1)+m(x)f(x) = 0$$

has a solution of the form (14), where the E's are determinate functions of x, and F(x) is an arbitrary periodic function of period one; then after at most n applications of the transformation S, we will have an equation of which the I invariant is zero, i.e.

$$I_{S_\mu}(x) = 0 \quad , \qquad (\mu \leqq n) .$$

To show this, substitute in (1) the expression for f(x) in (14), remembering that

$$F(x+\mu) = F(x).$$

We then have

$$E_0'(x+1)F(x)+E_0(x+1)F'(x)+ \text{-----} + E_n'(x+1)F^{(n)}(x)+E_n(x+1)F^{(n+1)}(x)$$
$$+p(x)\left[E_0'(x)F(x)+E_0(x)F'(x) + \text{-------} + E_n'(x)F^{(n)}(x)+E_n(x)F^{(n+1)}(x)\right]$$
$$+q(x)\left[E_0(x+1)F(x)+E_1(x+1)F'(x)+ \text{----} + E_n(x+1)F^{(n)}(x)\right]$$
$$+m(x)\left[E_0(x)F(x)+E_1(x)F'(x)+ \text{--------} + E_n(x)F^{(n)}(x)\right] = 0,$$

which may be written in the form

$$K_{n+1}(x)F^{(n+1)}(x)+K_n(x)F^{(n)}(x)+K_{n-1}(x)F^{(n-1)}(x)+ \text{------} = 0,$$

where

$$K_{n+1}(x) = E_n(x+1)+p(x)E_n(x),$$

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* Note that if $J_{S_n}(x) = 0$ then $I_{S_{n-1}}(x) = 0$, as may be seen by referring to § III.

$$K_n(x) = E_{n-1}(x+1) + p(x)E_{n-1}(x) + E_n'(x+1) + p(x)E_n'(x)$$
$$\qquad + q(x)E_n(x+1) + m(x)E_n(x),$$

$$K_{n-1}(x) = E_{n-2}(x+1) + p(x)E_{n-2}(x) + E_{n-1}'(x+1) + p(x)E_{n-1}'(x)$$
$$\qquad + q(x)E_{n-1}(x+1) + m(x)E_{n-1}(x+1),$$

and so forth. All of these K's must be zero since $f(x)$ satisfies the equation (1) for all values of $F(x)$. Hence

$$E_n(x+1) = -p(x)E_n(x),$$

and
$$E_{n-1}(x+1) + p(x)E_{n-1}(x) = \frac{d}{dx}\Big[p(x)E_n(x)\Big] - p(x)E_n'(x)$$
$$\qquad + p(x)q(x)E_n(x) - m(x)E_n(x)$$
$$= p'(x)E_n(x) + p(x)q(x)E_n(x) - m(x)E_n(x)$$
$$= -p(x)I(x)E_n(x).$$

Applying the transformation S to $f(x)$, we have

$$f_{S_1}(x) = f(x+1) + p(x)f(x)$$
$$= E_0(x+1)F(x) + E_1(x+1)F'(x) + \text{-----} + E_n(x+1)F^{(n)}(x)$$
$$+ p(x)\Big[E_0(x)F(x) + E_1(x)F'(x) + \text{----} + E_n(x)F^{(n)}(x)\Big]$$

$$= \Big[E_n(x+1) + p(x)E_n(x)\Big]F^{(n)}(x)$$
$$\qquad + \Big[E_{n-1}(x+1) + p(x)E_{n-1}(x)\Big]F^{(n-1)}(x) + \text{--------}$$

$$= 0 - p(x)I(x)F^{(n-1)}(x) + \text{----------} .$$

We see that the order of the transformed expression in $F(x)$ is less than before. Repeat the process, reducing the order each time, until we get one of the invariants $I_{S_k}(x)$ zero, or else we get a new dependent variable

$$f_{S_\mu}(x) = R(x)F(x)$$

which satisfies the equation

$$f_{S_\mu}'(x+1) + p_{S_\mu}(x)f_{S_\mu}'(x) + q_{S_\mu}(x)f_{S_\mu}(x+1) + m_{S_\mu}(x)f_{S_\mu}(x) = 0,$$

or $$R'(x+1)F(x) + R(x+1)F'(x) + p_{S_\mu}(x)\left[R'(x)F(x) + R(x)F'(x)\right] + q_{S_\mu}(x)R(x+1)F(x) + m_{S_\mu}(x)R(x)F(x) = 0.$$

This equation is an identity in $F(x)$, so the coefficients of $F(x)$ and of $F'(x)$ must be zero. Setting the coefficient of $F'(x)$ equal to zero, we get

$$p_{S_\mu}(x) = -\frac{R(x+1)}{R(x)}.$$

Putting this into the coefficient of $F(x)$ set equal to zero, we have

$$R'(x+1) - \frac{R(x+1)R'(x)}{R(x)} + q_{S_\mu}(x)R(x+1) + m_{S_\mu}(x)R(x) = 0.$$

Forming the invariant $I_{S_\mu}(x)$, we have

$$p(x)I_{S_\mu}(x) = m_{S_\mu}(x) - p_{S_\mu}(x)q_{S_\mu}(x) - p'_{S_\mu}(x)$$

$$\frac{R(x+1)R'(x)}{R^2(x)} - \frac{R'(x+1)}{R(x)} - q_{S_\mu}(x)\frac{R(x+1)}{R(x)} + q_{S_\mu}(x)\frac{R(x+1)}{R(x)} + \frac{R(x)R'(x+1) - R(x+1)R'(x)}{R^2(x)}$$

which is identically zero. Hence we see that $I_{S_\mu}(x) = 0$ is a necessary as well as a sufficient condition for a solution of the form

(14) $$f(x) = E_0(x)F(x) + E_1(x)F'(x) + \text{------} + E_n(x)F^{(n)}(x).$$

Suppose that $J_{T_n}(x) = 0$* and $J_{T_k}(x) \neq 0$, where k takes the values 0,1,2,3,------,n-1. The equation in $f_{T_n}(x)$ can then be written

$$f'_{T_n}(x+1) + q_{T_n}(x)f_{T_n}(x+1) - \left[-p(x)\right]\left[f'_{T_n}(x) + q_{T_n}(x-1)f_{T_n}(x)\right] = 0,$$

which, as was shown in § II, has a solution of the form

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* Note that if $I_{T_n}(x) = 0$, then $J_{T_{n-1}}(x) = 0$, as may be seen by referring to § III.

$$f_{T_n}(x) = e^{-\int_{x_0}^{x} q_{T_n}(x-1)dx} \int_{x_0}^{x} \theta(x)\, e^{\sum \log[-p(x)] + \int_{x_0}^{x} q_{T_n}(x-1)dx} + Ke^{-\int_{x_0}^{x} q_{T_n}(x-1)dx}$$

where $\theta(x)$ is an arbitrary function of period one, and K is an arbitrary constant. We will write for the sake of brief notation

$$e^{-\sum \log[-p(x)]} f_{T_n}(x) = e^{\lambda(x)} V(x).$$

We have already developed the formula

$$(13) \qquad e^{-\sum \log[-p(x)]} f(x) = C_0 \Delta\{C_1 \Delta [\text{--------} \Delta ( C_n \Delta D_n )]\}.$$

In our present case

$$D_n = e^{\lambda(x)} V(x), \quad \text{and} \quad C_n = \frac{-1}{p(x) J_{T_{n-1}}(x)}$$

Then f(x) takes the form

$$(15) \qquad f(x) = W_0(x)V(x) + W_1(x)V(x+1) + \text{--------} + W_n(x)V(x+n),$$

where the W's are determinate functions. Choosing $\theta(x) = 0$, we have the simpler integral

$$f(x) = K\left[W_0(x) + W_1(x) + \text{--------} + W_n(x)\right].$$

or $\qquad f(x) = K\, W(x), \qquad\qquad (16)$

where $W(x)$ is a determinate function and K is an arbitrary constant

The solution (15) we will call of rank $n+1$ of the <u>second kind</u>, and the equation for which $J_{T_n}(x) = 0$ and $J_{T_k}(x) \neq 0$, where k takes the values $0, 1, 2, 3, \text{-------}, n-1$, we will also call of rank $n+1$ of the second kind. In this paper, we will be concerned with the rank of the equation rather than with the rank of the solution.

## VI. EQUATIONS OF FINITE RANK OF THE FIRST KIND.

Suppose the equation (1) is of rank $n+1$ of the first kind. We then have

$$\alpha_{S_n}(x) = m_{S_n}(x) - p_{S_n}(x)q_{S_n}(x) - p'_{S_n}(x) = 0.$$

If $p_{S_n}(x)$ and $q_{S_n}(x)$ are chosen arbitrarily, $m_{S_n}(x)$ is defined by this equation. Using the expression for $m_{S_n}(x)$ thus defined, the invariant

$$J_{S_n}(x) = \frac{m_{S_n}(x)}{p_{S_n}(x)} - q_{S_n}(x-1)$$

becomes

$$J_{S_n}(x) = \Delta q_{S_n}(x) + \frac{d}{dx} \log p_{S_n}(x).$$

We have proved that

$$I_{S_{m+1}}(x) = I_{S_m}(x+1) + I_{S_m}(x) - J_{S_m}(x) - \Delta\frac{d}{dx} \log I_{S_m}(x),$$

and $\quad J_{S_{m+1}}(x) = I_{S_m}(x),$

from which we get

$$J_{S_{m-1}}(x) = J_{S_m}(x+1) + J_{S_m}(x) - I_{S_m}(x) - \Delta \frac{d}{dx} \log J_{S_m}(x).$$

Therefore, since $I_{S_n}(x) = 0$,

$$J_{S_{n-1}}(x) = J_{S_n}(x+1) + J_{S_n}(x) - \Delta \frac{d}{dx} \log J_{S_n}(x).$$

So we can now calculate the coefficients in backward succession.

Since $q_{S_{m+1}}(x) = q_{S_m}(x-1)$,

we have

$$q(x) = q_{S_n}(x-n).$$

Also we have

$$p_{S_{n-1}}(x+1) = p_{S_n}(x) \frac{I_{S_{n-1}}(x)}{I_{S_{n-1}}(x+1)},$$

or
$$p_{S_{n-1}}(x+1) = p_{S_n}(x)\,\frac{J_{S_n}(x)}{J_{S_n}(x+1)},$$

also
$$p_{S_{n-2}}(x+2) = p_{S_{n-1}}(x+1)\,\frac{J_{S_{n-1}}(x+1)}{J_{S_{n-1}}(x+2)}$$
$$= p_{S_n}(x)\,\frac{J_{S_n}(x)J_{S_{n-1}}(x+1)}{J_{S_n}(x+1)J_{S_{n-1}}(x+2)},$$

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$$p(x+n) = p_{S_n}(x)\,\frac{J_{S_n}(x)J_{S_{n-1}}(x+1)\text{---------}J_{S_1}(x+n-1)}{J_{S_n}(x+1)J_{S_{n-1}}(x+2)\text{------}J_{S_1}(x+n)}.$$

Reducing the argument by n ,we have

$$p(x) = p_{S_n}(x-n)\,\frac{J_{S_n}(x-n)J_{S_{n-1}}(x-n+1)\text{----------}J_{S_1}(x-1)}{J_{S_n}(x-n+1)J_{S_{n-1}}(x-n+2)\text{---------}J_{S_1}(x)}.$$

Also we have
$$m(x) = p(x)q(x) + p'(x) + p(x)I(x)$$
$$= p(x)q(x) + p'(x) + p(x)J_{S_1}(x),$$

or we may use the form
$$m(x) = p(x)q(x-1) + p(x)J(x).$$

Thus we have developed the restrictions upon the coefficients of (1) which must exist if (1) is of finite rank of the first kind. That these conditions are also sufficient, may be seen by reversing the steps of the discussion.

## VII. EQUATIONS OF FINITE RANK OF THE SECOND KIND.

Suppose the equation (1) is of rank $n+1$ of the second kind. Then we have

$$\beta_{T_n}(x) = m_{T_n}(x) - p_{T_n}(x)q_{T_n}(x-1) = 0.$$

Hence, if $p_{T_n}(x)$ and $q_{T_n}(x)$ are arbitrarily given, $m_{T_n}(x)$ is determined. We have already found

$$J_{T_n}(x) = J_{T_{n-1}}(x) + J_{T_{n-1}}(x-1) - I_{T_{n-1}}(x-1) - \Delta \frac{d}{dx} \log J_{T_{n-1}}(x-1),$$

and $\qquad J_{T_{n-1}}(x) = I_{T_n}(x).$

From these we get, rembering that $J_{T_n}(x) = 0$,

$$I_{T_{n-1}}(x-1) = I_{T_n}(x) + I_{T_n}(x-1) - \Delta \frac{d}{dx} \log I_{T_n}(x-1),$$

and $\qquad I_{T_n}(x) = J_{T_{n-1}}(x) = \dfrac{m_{T_n}(x)}{p_{T_n}(x)} - q_{T_n}(x) - \dfrac{p'_{T_n}(x)}{p_{T_n}(x)}$

$$= -\Delta q_{T_n}(x-1) - \frac{p'_{T_n}(x)}{p_{T_n}(x)} .$$

Now we can calculate the coefficients in backward succession. We have seen that

$$p(x) = p_{T_n}(x).$$

We also have

$$q_{T_n}(x) = q_{T_{n-1}}(x-1) - \frac{d}{dx} \log\left[J_{T_{n-1}}(x)p(x)\right] .$$

whence

$$q_{T_{n-1}}(x-1) = q_{T_n}(x) + \frac{d}{dx} \log\left[J_{T_{n-1}}(x)p(x)\right] .$$

Hence it follows that

$$q_{T_{n-2}}(x-2) = q_{T_n}(x) + \frac{d}{dx} \log\left[J_{T_{n-1}}(x)J_{T_{n-2}}(x-1)p(x)p(x-1)\right] ,$$

$$q_{T_{n-3}}(x-3) = \text{------------} \quad \text{--------}$$

$$\text{--------} \quad \text{-------} \quad \text{-----} \quad \text{----} \quad \text{-----}$$

$$q(x-n) = q_{T_n}(x) + \frac{d}{dx} \log\left[J_{T_{n-1}}(x)\text{-----}J(x-n)p(x)\text{----}p(x-n)\right].$$

Increasing the argument by n,we have

$$q(x) = q_{T_n}(x+n) + \frac{d}{dx} \log\left[J_{T_{n-1}}(x+n)\text{-----}J(x)p(x+n)\text{-----}p(x)\right].$$

or
$$q(x) = q_{T_n}(x+n) + \frac{d}{dx} \log\left[I_{T_n}(x+n)\text{-----}I_{T_1}(x)p(x+n)\text{-----}p(x)\right].$$

To determine m(x),we have

$$m(x) = p(x)q(x-1) + p(x)J(x),$$

or
$$m(x) = p(x)q(x-1) + p(x)I_{T_1}(x).$$

Thus we have found the restrictions on the coefficients of (1) which must hold if (1) is of finite rank of the second kind. By reversing the steps of the discussion we may see that these conditions are also sufficient. So we now have the necessary and sufficient conditions that (1) shall be of finite rank of either the first or the second kind.

## VIII. EQUATIONS OF DOUBLY FINITE RANK.

The equation (1) is said to be of doubly finite rank when it is of finite rank with respect to both S and T. Suppose it is of rank $k+1$ with respect to T, and of finite rank with respect to S. Transforming it k times by T, we have, since $J_{T_k}(x) = 0$,

(16) $$f'_{T_k}(x+1)+p(x)f'_{T_k}(x)+q_{T_k}(x)f_{T_k}(x+1)+p(x)q_{T_k}(x-1)f_{T_k}(x) = 0.$$

This equation is also of finite rank with respect to S. Suppose that rank is $r+1$. We wish to see what restrictions are then imposed upon the coefficients $p(x)$ and $q_{T_k}(x)$.

First, apply the transformation

$$f_{T_k}(x) = v(x)h(x) ,$$

which does not change the rank. We will choose $v(x)$ so that the coefficient of $h(x+1)$ in the new equation shall be zero. This requires that

$$\frac{v'(x+1)}{v(x+1)} = - q_{T_k}(x),$$

*whence* $$v(x) = e^{-\int_{x_0}^{x} q_{T_k}(x-1)dx} .$$

The equation then becomes

$$h'(x+1)+p(x)\frac{v(x)}{v(x+1)}h'(x) = 0,$$

which may be written in the form

(17) $$h'(x+1)+p(x)e^{\int_x^{x+1} q_{T_k}(x-1)dx}\, h'(x) = 0.$$

This equation, being of rank $r+1$ with respect to S, has a solution

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of the form

$$h(x) = E_0(x)F(x) + E_1(x)F'(x) + \text{--------} + E_r(x)F^{(r)}(x),$$

where $F(x)$ is an arbitrary function of period one. Then $h'(x)$ has the form

$$h'(x) = Z_0(x)F(x) + Z_1(x)F'(x) + \text{-------} + Z_{r+1}(x)F^{(r+1)}(x),$$

where $\quad Z_0(x) = E_0'(x), \qquad Z_1(x) = E_0(x) + E_1'(x), \qquad \text{------}$.

Substitute this value of $h'(x)$ in (17). Since it must satisfy (17) identically, we have the relations

(18) $\quad Z_i(x+1) - R(x)Z_i(x) = 0,$

where $R(x)$ is the negative of the coefficient of $h'(x)$ in (17).

Let $\bar{Z}(x)$ be any particular solution of (18). The other solutions may then be expressed in the form

$$Z_i(x) = w_i(x)\bar{Z}(x),$$

where the functions $w_i(x)$ have the period one. In particular

$$Z_0(x) = w_0(x)\bar{Z}(x) = E_0'(x),$$

$$Z_1(x) = w_1(x)\bar{Z}(x) = E_0(x) + E_1'(x),$$

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$$Z_r(x) = w_r(x)\bar{Z}(x) = E_{r-1}(x) + E_r'(x),$$

$$Z_{r+1}(x) = w_{r+1}(x)\bar{Z}(x) = E_r(x).$$

From these we get

$$E_r(x) = w_{r+1}(x)\bar{Z}(x),$$

$$E_{r-1}(x) = w_r(x)\bar{Z}(x) - \frac{d}{dx}\left[w_{r+1}(x)\bar{Z}(x)\right],$$

$$E_{r-2}(x) = w_{r-1}(x)\bar{Z}(x) - \frac{d}{dx}\left[w_r(x)\bar{Z}(x)\right] + \frac{d^2}{dx^2}\left[w_{r+1}(x)\bar{Z}(x)\right],$$

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$$E_{r-3}(x) = w_{r-2}(x)\bar{Z}(x) - \frac{d}{dx}\left[w_{r-1}(x)\bar{Z}(x)\right] + \frac{d^2}{dx^2}\left[w_r(x)\bar{Z}(x)\right] - \frac{d^3}{dx^3}\left[w_{r+1}(x)\bar{Z}(x)\right]$$

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$$E_0(x) = w_1(x)\bar{Z}(x) - \frac{d}{dx}\left[w_2(x)\bar{Z}(x)\right] + \text{----------}$$

$$+ \text{------} + (-1)^r \frac{d^r}{dx^r}\left[w_{r+1}(x)\bar{Z}(x)\right].$$

Since $w_0(x)\bar{Z}(x) = E'(x)$, we have

$$w_0(x)\bar{Z}(x) = \frac{d}{dx}\left[w_1(x)\bar{Z}(x)\right] - \text{------} + (-1)^r \frac{d^{r+1}}{dx^{r+1}}\left[w_{r+1}(x)\bar{Z}(x)\right],$$

which may be written in the form

$$(19) \qquad \bar{Z}^{(r+1)}(x) + \eta_1(x)\bar{Z}^{(r)}(x) + \text{---------} + \eta_{r+1}(x)\bar{Z}(x) = 0.$$

The steps in the foregoing discussion are reversible. Hence it follows that the necessary and sufficient condition that the equation (1) shall be of doubly finite rank is that

$$p(x)\, e^{\int_x^{x+1} q_{T_K}(x-1)dx} = \frac{\bar{Z}(x+1)}{\bar{Z}(x)},$$

where $\bar{Z}(x)$ is a solution of equation (19), in which the coefficients $\eta_i(x)$ involve arbitrary periodic functions as shown above.

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## IX. THE ANALOGUES OF LÉVY'S TRANSFORMATIONS.

In this section we consider generalizations of the Laplace-Poisson transformations, and investigate their usefulness in obtaining equations with vanishing invariants. These transformations are analogous to those applied by Lévy* to the analogous partial differential equation. They are

$$G(x) = f(x+1) + \left[p(x) + \gamma(x)\right] f(x),$$

and $$H(x) = f'(x) + \left[q(x-1) + \delta(x)\right] f(x).$$

The first of these transforms the equation (1) into

$$G'(x+1) + p_G(x) G'(x) + q_G(x) G(x+1) + m_G(x) G(x) = L(x) f(x),$$

where $$p_G(x) = p(x)\,\frac{\gamma(x+1)}{\gamma(x)},$$

$$q_G(x) = q(x+1),$$

$$m_G(x) = p(x)q(x)\,\frac{\gamma(x+1)}{\gamma(x)} + \alpha(x+1) - \gamma'(x+1) - \gamma(x+1)\Delta q(x),$$

and $$L(x) = \left[p(x)+\gamma(x)\right]\left[\alpha(x+1) - \gamma'(x+1) - \gamma(x+1)\Delta q(x)\right] + \frac{\gamma(x+1)}{\gamma(x)}\left\{\left[-\alpha(x) + \gamma'(x) + \gamma(x)q(x)\right] p(x) - m(x)\gamma(x)\right\}$$

where $\alpha(x)$ is, as before, $m(x) - p(x)q(x) - p'(x)$.

Let $\gamma(x)$ be so chosen that $L(x) = 0$. Call the invariants of the resulting equation $I_G(x)$ and $J_G(x)$, and the corresponding relative invariants $\alpha_G(x)$ and $\beta_G(x)$.

Suppose that $J_G(x) = 0$. This condition and the condition $L(x) = 0$ together give

*Jour. de l'École Polytechnique, t.38(1886), p.67.

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$$p(x)\left[-\alpha(x)+\gamma'(x)+\gamma(x)q(x)\right] - m(x)\gamma(x) = 0,$$

or, since

$$\alpha(x) - \gamma'(x) - \gamma(x)\Delta q(x-1) = 0,$$

$$\gamma(x)q(x-1)p(x) - m(x)\gamma(x) = 0.$$

The case when $\gamma(x) = 0$ is that of the transformation S. If $\gamma(x) \neq 0$, we have $q(x-1)p(x) - m(x) = 0$,

or $\quad J(x) = 0.$

Hence the transformed equation cannot have a vanishing J-invariant unless we have $J(x) = 0$ from the original equation, and therefore this transformation is of no value in securing an equation with a vanishing J-invariant.

If $I_G(x) = 0$, we have

$$\begin{aligned}\alpha_G(x) = \; & \alpha(x+1) - \gamma'(x+1) + q(x)\gamma(x+1) - q(x+1)\gamma(x+1) \\ & + p(x)q(x)\frac{\gamma(x+1)}{\gamma(x)} + p(x)q(x+1)\frac{\gamma(x+1)}{\gamma(x)} \\ & - p'(x)\gamma(x+1)\gamma(x) - p(x)\frac{\gamma'(x+1)}{\gamma(x)} + p(x)\gamma(x+1)\frac{\gamma'(x)}{\gamma^2(x)} \\ = \; & 0.\end{aligned}$$

Set $\alpha_G(x)$ equal to $L(x)$, which we require to be also zero, and clear of fractions. The condition reduces to

$$\gamma(x)\gamma(x+1)\alpha(x) + p(x)\gamma(x+1)\alpha(x) - p(x)\gamma(x)\alpha(x+1) = 0,$$

which we write in the form

$$(20) \qquad \gamma(x+1) = \frac{p(x)\alpha(x+1)\gamma(x)}{\alpha(x)\gamma(x) + p(x)\alpha(x)},$$

assuming $\alpha(x) \neq 0$. Substituting (20) into the equation $L(x) = 0$, we get an equation which has as solutions: $\gamma(x) = 0$, which is the

case of the transformation S, and

$$\gamma(x) = \frac{\Omega(x)}{\Theta(x)},$$

where

$$\Omega(x) = \alpha(x)\alpha(x+1)p(x)m(x) - \alpha(x)\alpha(x+1)p^2(x)q(x) \\ -2\alpha(x+1)\alpha(x)p(x) + \alpha(x)\alpha'(x+1)p^2(x) \\ -\alpha(x+1)\alpha'(x)p^2(x) + \alpha(x)\alpha(x+1)p^2(x)\Delta q(x),$$

and $$\Theta(x) = \alpha^2(x+1)\alpha'(x) - \alpha(x)\alpha(x+1)p'(x) - \alpha(x)\alpha'(x+1)p(x) \\ -\alpha(x)\alpha(x+1)p(x)\Delta q(x) + \alpha(x+1)\alpha'(x)p(x).$$

This condition must be satisfied simultaneously with (20). Putting this value of $\gamma(x)$ into the right hand member of (20), we have,

$$\gamma(x+1) = \frac{\alpha(x+1)p(x)\Omega(x)}{\alpha(x)\Omega(x) + \dot{\alpha}(x)p(x)\Theta(x)}$$

Hence we must have

$$\frac{\Omega(x+1)}{\Theta(x+1)} = \frac{\alpha(x+1)p(x)\Omega(x)}{\alpha(x)\Omega(x) + \dot{\alpha}(x)p(x)\Theta(x)}.$$

That this equality is not satisfied in general, is shown by the fact that it is not satisfied in case of the equation

$$f'(x+1) + f'(x) + xf(x+1) + 2xf(x) = 0.$$

Here $p(x) = 1$, $q(x) = x$, $m(x) = 2x$, and $\alpha(x) = 2x-x-0 = x$. So the condition becomes

$$\frac{x^3+3x^2+4x+3}{x} = \frac{1-x^2-x^3-x^4}{2x-2x^2-x^4},$$

which is not an identity.

We have now established that the transformation

$$G(x) = f(x+1) + \left[p(x) + \gamma(x)\right] f(x),$$

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where $\gamma(x) \neq 0$, is generally not of value in obtaining an equation with a vanishing invariant.

Making the transformation

$$H(x) = f'(x) + \left[q(x-1) + \delta(x)\right] f(x),$$

the equation (1) becomes

$$H'(x+1) + p_H(x)H'(x) + q_H(x)H(x+1) + m_H(x)H(x) = Y(x)f(x),$$

where

$$p_H(x) = p(x),$$

$$q_H(x) = q(x) - \frac{\delta'(x+1)}{\delta(x+1)},$$

$$m_H(x) = p(x) + \beta(x) + p(x)\,\Delta\delta(x) + p(x)\left\{q(x) - \frac{\delta'(x+1)}{\delta(x+1)}\right\},$$

and

$$\begin{aligned} Y(x) = & -\beta(x)q(x) + \beta(x)\frac{\delta'(x+1)}{\delta(x+1)} + \beta(x)q(x-1) - \beta(x)\,\delta(x+1) \\ & -\beta'(x) + \beta(x)\,\delta(x) + \delta(x)p(x)q(x) - \delta(x)p(x)\frac{\delta'(x+1)}{\delta(x+1)} \\ & -p(x)\,\delta(x)q(x+1) + p(x)\delta(x)\delta(x+1) \\ & + p'(x)\delta(x) + p(x)\,\delta'(x) - p(x)\,\delta^2(x). \end{aligned}$$

If $Y(x) = 0$, we have an equation of the same form as (1). Let $I_H(x)$ and $J_H(x)$ denote the invariants of this equation, and $\alpha_H(x)$ and $\beta_H(x)$ the corresponding relative invariants.

Suppose $I_H(x) = 0$; then $\alpha_H(x) = 0$. Equating $\alpha_H(x)$ to $Y(x)$, which we assume to be also zero, we have a condition on $\delta(x)$ which reduces to

$$\beta(x) + p(x)\Delta\delta(x) = 0,$$

or

$$\delta(x+1) = \delta(x) - J(x).$$

Substituting this into the condition $Y(x) = 0$, we have

$$\left[J(x) - \delta(x)\right]\left[-p(x)\Delta q(x-1)+\beta(x) - p'(x)\right] = 0,$$

which may be reduced to

$$\delta(x+1)\alpha(x) = \delta(x+1)p(x)I(x) = 0.$$

Since $p(x) \neq 0$, we must have either $\delta(x+1) = 0$, in which case the transformation is simply T, or we must have $I(x) = 0$, in which case the transformation is not needed. So this transformation is of no value in securing an equation with a vanishing I-invariant.

Suppose $J_H(x)$ is 0 when $Y(x)$ is 0. Equate these two expressions and clear of fractions. The condition reduces to

$$\beta(x)\,\delta'(x+1) = \beta'(x)\delta(x+1)+\beta(x)q(x)\delta(x+1)-\beta(x)q(x-1)\delta(x+1)+\beta(x)\delta^2(x+1)$$

or

(21) $$\delta'(x+1) = \left\{\frac{\beta'(x)}{\beta(x)}+\Delta q(x-1)+\delta(x+1)\right\}\delta(x+1).$$

Substituting (21) into the equation $Y(x) = 0$, and reducing, we get

$$\delta(x)\left[\beta(x) - p(x)\Delta\left(q(x-2) - \frac{\beta'(x-1)}{\beta(x-1)}\right)+p'(x)\right] = 0 .$$

Hence we must have either $\delta(x) = 0$, which is the case of the transformation T, or else the quantity in the brackets equal to zero. That this quantity is not in general zero, may be seen by considering the special case where $p(x) = 1, q(x) = x$, and $m(x) = 2x$. The quantity in the brackets then becomes

$$x - \frac{2}{(x-1)(x-2)},$$

which is not identically zero.

So we have the result that the two transformations investigated in this section are not generally useful in obtaining an equation with a vanishing invariant.

## VITA

Raymond Franklin Borden.

Born at Portsmouth,R.I. on Aug.2,1892. Received elementary instruction at home,and at school in District No.2,Portsmouth,until 1904. Graduated from Calvert Grammer School,Newport,R.I. in 1905, and from Rogers High School,Newport,in 1909. Entered Brown University in 1910,and graduated with degree of Ph.B. in 1914. Elected to Phi Beta Kappa and to Sigma Xi,and also appointed as a "Francis Wayland Scholar" in 1914. Received the degree of A.M. from Brown in 1915 for graduate work in Mathematics and Physics. Also assisted in the Mathematics Department at Brown,and taught Mathematics in the evening school of the Rhode Island School of Design,in 1914-15. Came to the University of Illinois in 1915 as Assistant in Mathematics and held that position through 1915-16,1916-17,1917-18,and in the summer of 1917.

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University of Illinois, R.F.Borden.

May 1918.

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